1. Let l_1 be the line through the point $(3,2,-1)$ and $(1,-1,2)$

Let l_2 be the line

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z}{-2}$$

$$\text{Direction of } l_1 = (3,2,-1) - (1,-1,2)$$
$$= (2,3,-3)$$

$$\text{Direction of } l_2 = (3,2,-2)$$

Equation of the required plane is

$$\begin{vmatrix} x-3 & y-2 & z+1 \\ 2 & 3 & -3 \\ 3 & 2 & -2 \end{vmatrix} = 0$$

$$-5(y-2) + (-5)(z+1) = 0$$
$$y + z = 1$$

2. Let L be the line

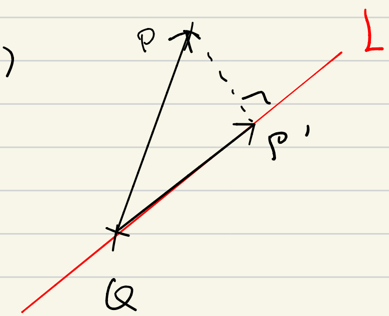
$$\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$$

$Q(1,-1,0)$ is a point on the line L.

$$\overrightarrow{QP} = \overrightarrow{OP} - \overrightarrow{OQ}$$

$$= (2,0,2)$$

$$\overrightarrow{QP'} = \frac{\overrightarrow{QP} \cdot (2,-1,3)}{|(2,-1,3)|^2}(2,-1,3)$$



$$= \frac{10}{14}(2,-1,3)$$

$$= \frac{5}{7}(2,-1,3)$$

$$\overrightarrow{PP'} = \overrightarrow{QP'} - \overrightarrow{QP}$$

$$= \left(-\frac{4}{7}, -\frac{5}{7}, \frac{1}{7}\right)$$

$\therefore$ The line through P and perpendicular to L is $(3,-1,2) + t(-4,-5,1)$, $t \in \mathbb{R}$

3. Distance between P and the plane is

$$\left| \frac{2(-1) - 3(3) + 4(2) - 5}{\sqrt{2^2 + 3^2 + 4^2}} \right| = \frac{8}{\sqrt{29}}$$

$\therefore$ The radius is $\frac{8}{\sqrt{29}}$

and equation of the sphere is

$$(x+1)^2 + (y-3)^2 + (z-2)^2 = \frac{64}{29}$$

4. For each constant k, the equation

$$A_1 x + B_1 y + C_1 z + D_1 + k(A_2 x + B_2 y + C_2 z + D_2) = 0$$

represents a plane passing through the line of intersection of the given two planes.

Verification :

① Clearly, the equation is equivalent to

$$(A_1 + kA_2)x + (B_1 + kB_2)y + (C_1 + kC_2)z + (D_1 + kD_2) = 0,$$

It is a plane with normal vector

$$(A_1 + kA_2,\ B_1 + kB_2,\ C_1 + kC_2)$$

This vector is nonzero because the two planes intersect along a line.

Consider the system of equations:

$$\left(\begin{array}{ccc|c} A_1 & B_1 & C_1 & -D_1 \\ A_2 & B_2 & C_2 & -D_2 \end{array}\right)$$

The solution set is spanned by one vector.

$\therefore$ The matrix $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ has rank 2

(A_1, B_1, C_1) and (A_2, B_2, C_2) are linearly independent.

(2) The generated plane contains the line of intersection: Suppose (x_0, y_0, z_0) belongs to the line, then it satisfies

$$A_1x_0 + B_1y_0 + C_1z_0 + D_1 = 0 \quad \text{and}$$

$$A_2 x_0 + B_2 y_0 + C_2 z_0 + D_2 = 0$$

Then it clearly also satisfies

$$(A_1 x_0 + B_1 y_0 + C_1 z_0 + D_1) + k(A_2 x_0 + B_2 y_0 + C_2 z_0 + D_2) = 0.$$

Additional Observation :

Using the point - plane distance formula,

for $(x, y, z) \in \mathbb{R}^3$

Distance between (x, y, z) and plane 1 is

$$\left| \frac{A_1 x + B_1 y + C_1 z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} \right|$$

Distance between (x, y, z) and plane 2 is

$$\left| \frac{A_2 x + B_2 y + C_2 z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}} \right|$$

For the new plane ,

$$A_1 x + B_1 y + C_1 z + D_1 + k(A_2 x + B_2 y + C_2 z + D_2) = 0$$

It is equivalent to

$$\frac{A_1 x + B_1 y + C_1 z + D_1}{\sqrt{A_1^2 + B_1^2 + C_1^2}} = -k \frac{\sqrt{A_2^2+B_2^2+C_2^2}}{\sqrt{A_1^2+B_1^2+C_1^2}} \cdot \frac{A_2 x + B_2 y + C_2 z + D_2}{\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

By taking absolute value on both sides,

a simple observation is that

All points in the plane satisfies

$$\frac{\text{Distance from plane 1}}{\text{Distance from plane 2}} = |k| \cdot \frac{\sqrt{A_2^2+B_2^2+C_2^2}}{\sqrt{A_1^2+B_1^2+C_1^2}}$$

(RHS is a constant number when k is fixed.)

5. Consider the augmented matrix

$$\left(\begin{array}{ccc|c} 7 & -2 & -2 & 5 \\ 3 & 2 & -3 & 10 \\ 7 & 2 & -5 & 16 \end{array}\right)$$

$$\xrightarrow[-2R_2+R_1]{-R_1+R_3} \left(\begin{array}{ccc|c} 1 & -6 & 4 & -15 \\ 3 & 2 & -3 & 10 \\ 0 & 4 & -3 & 11 \end{array}\right)$$

$$\xrightarrow{-3R_1+R_2} \left(\begin{array}{ccc|c} 1 & -6 & 4 & -15 \\ 0 & 20 & -15 & 55 \\ 0 & 4 & -3 & 11 \end{array}\right)$$

$$\xrightarrow[\frac{1}{4}R_2]{-5R_3+R_2} \left(\begin{array}{ccc|c} 1 & -6 & 4 & -15 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{3}{4} & \frac{11}{4} \end{array}\right)$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{6R_3+R_1} \left(\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 1 & -\frac{3}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ The solution set is

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : \begin{array}{l} x = \frac{3}{2} + \frac{1}{2}z \\ y = \frac{11}{4} + \frac{3}{4}z \end{array}, \ z \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} \frac{3}{2} \\ \frac{11}{4} \\ 0 \end{pmatrix} + z \begin{pmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 1 \end{pmatrix} : z \in \mathbb{R} \right\}$$

Plugging in $z = 3$, we see that it is a straight line passing through $(3, 5, 3)$ with direction $\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

6. From the question, we know that

① $\vec{r}(0) = (1, 2, 1)$

② $\vec{a}(t) = (40\cos 45^\circ, 40\sin 45^\circ, -32)$

$= (20\sqrt{2}, 20\sqrt{2}, -32)$

③ $\vec{v}(0) = (60\cos 60^\circ, 0, 60\sin 60^\circ)$

$= (30, 0, 30\sqrt{3})$

where $\vec{v}(t) = \vec{r}\,'(t)$, $\vec{a}(t) = \vec{r}\,''(t)$

$$\vec{v}(t) - \vec{v}(0) = \int_0^t \vec{a}(s)\,ds$$

$$= (20\sqrt{2}\,t, 20\sqrt{2}\,t, -32t)$$

$$\vec{v}(t) = (20\sqrt{2}\,t, 20\sqrt{2}\,t, -32t) + (30, 0, 30\sqrt{3})$$

$$\vec{r}(t) - \vec{r}(0) = \int_0^t \vec{v}(s)\,ds$$

$$= (10\sqrt{2}\,t^2, 10\sqrt{2}\,t^2, -16t^2) + (30t, 0, 30\sqrt{3}\,t)$$

$\therefore$ $\vec{r}(t) = t^2(10\sqrt{2}, 10\sqrt{2}, -16) + t(30, 0, 30\sqrt{3}) + (1, 2, 1)$

7. Take A as a reference point.

Volume of the parallelepiped form by A, B, C, D

is $|\vec{AB} \cdot (\vec{AC} \times \vec{AD})|$

$\therefore$ Four points are coplanar iff

$$\vec{AB} \cdot (\vec{AC} \times \vec{AD}) = 0$$